To achieve a characterization of DoH's, we introduce

<u>Def.</u> Let $K \subset\subset \Omega$. The <u>$\mathcal{O}(\Omega)$-hull</u> $\hat{K}_\Omega$ of K is given by

$$\hat{K}_\Omega = \{ z \in \Omega : |u(z)| \leq \sup_K |u| \}$$

<u>Thm 3</u> Let $\Omega \subseteq \mathbb{C}^n$ domain. TFAE:

(i) Ω is a DoH.

(ii) $K \subset\subset \Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$ and

$$d(K, \partial\Omega) = d(\hat{K}_\Omega, \partial\Omega)$$

distance to $\partial\Omega$: $\inf\limits_{z \in K, w \in \partial\Omega} |z-w|$ (or any other distance)

(iii) $\exists\, u \in \mathcal{O}(\Omega)$ that cannot be extended past Ω, i.e. $\nexists\, \Omega_1, \Omega_2$ as in DoH definition for u.

<u>Sketch of pf.</u> Clearly, (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). We prove something slightly different but which can be used to deduce (ii). Let D be a fixed polydisk centered at 0 of some polyradius $r=(r_1,\dots,r_n)$. Let $\Delta^D_\Omega(z) := \sup\{s>0 : \{z\}+sD \subseteq \Omega\}$. (This is some measure of distance from z to Ω^c.)

We shall show that $\inf_{z\in K} \Delta^D_\Omega(z) = \inf_{z\in \hat{K}_\Omega} \Delta^D_\Omega(z)$.

This $\Rightarrow$ $\hat{K}_\Omega \subset\subset \Omega$ and can be used to show the full (ii). Since $K \subset\subset \Omega$, $\delta := \inf_K \Delta^D_\Omega > 0$.

Choose $u \in \mathcal{O}(\Omega)$, for any $t<1 \Rightarrow \overline{\{z\}+t\delta D} \subseteq \Omega$ for all $z\in K$ and $|u| \le M_t$ for some $M_t > 0$.

By Cauchy Estimates (that follow from the CIF in a polydisk in the same way as in $\mathbb{C}$), we then have

$$\left|\frac{\partial^\alpha u}{\partial z^\alpha}(z_0)\right| \le \frac{M_t\,\alpha!}{(t\delta)^{|\alpha|}\, r^\alpha}, \quad \forall z_0\in K$$

By def. of $\hat{K}_\Omega$, the same estimates hold for all $z \in \hat{K}_\Omega$.

But these estimates also show that the P.S.

$$u(z) = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha u}{\partial z^\alpha}(z_0)(z-z_0)^\alpha \qquad (*)$$

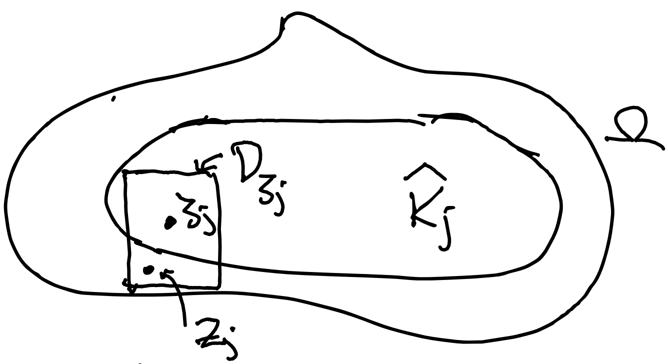
converges normally in the polydisk $\{z_0\} + t\delta D$ for any $z_0 \in \hat{K}_\Omega$. Since $t<1$ was arbitrary, we conclude that the P.S. $(*)$ conv. normally in $\{z_0\}+\delta D$ for any $z_0 \in \hat{K}_\Omega$. If there was $z_0 \in \hat{K}_\Omega$ s.t. $\Delta^D_\Omega(z_0) < \delta$ then the open polydisk $\{z_0\}+\delta D$ would not be contained in Ω (sticks out) so we could take $\Omega_2 = \{z_0\}+\delta D$, $\Omega_1 = (\{z_0\}+\delta D) \cap \Omega \neq \emptyset$ in def. of DoH to contradict Ω being a DoH since every u extends (by its normally convergent power series) to $\{z_0\}+\delta D$. Next:

(ii) (in fact, only $\hat{K}_\Omega \subset\subset \Omega$) $\Rightarrow$ (iii)

Let D be a given polydisk centered at 0, as above.

For each $\zeta \in \Omega$, $D_\zeta = \{\zeta\} + \Delta_\Omega^D(\zeta) D$ denotes the largest polydisk of "shape" D centered at ζ and contained in Ω. Let $A \subseteq \Omega$ be a countable dense set, $\{\zeta_j\}_{j=1}^\infty$ a seq. that contains each $a \in A$ infinitely many times. Let $K_1 \subseteq K_2 \subseteq \dots \subset\subset \Omega$ be an exhaustion of Ω by compacts. The assumption is that each hull $\hat{K}_j \subset\subset \Omega$.

For each j, let $z_j \in D_{\zeta_j}$ s.t. $z_j \notin \hat{K}_j$.



Since $z_j \notin \hat{K}_j$ $\exists$ $f_j \in \mathcal{O}(\Omega)$ s.t. $|f_j(z_j)| = 1$ and $\sup_{K_j} |f_j| < 1$. By replacing f_j by $f_j^{\text{high power}}$,

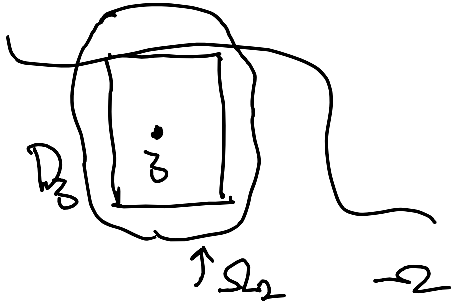
we may assume WLOG $\sup_{K_j} |f_j| < \frac{1}{2^j}$

Consider $f = \prod_{j=1}^{\infty}(1-f_j)^j$. For any $K \subset\subset \Omega$, $K \subseteq K_j$ for $j \geq N \Rightarrow$ For $j \geq N$ $|f_j| \leq \frac{1}{2^j} \Rightarrow \sum_{j=1}^{\infty} j|f_j| < \infty$ on $K \Rightarrow$ $f = \prod_{j=1}^{\infty}(1-f_j)^j$ conv. unif. on K and $f(z)=0 \Leftrightarrow f_j(z)=1$ for some $j \Rightarrow f \not\equiv 0$.

(This is proved in 220B; see Conway).

Since $f_j(z_j)=1 \Rightarrow \partial^\alpha(1-f_j)^j\Big|_{z=z_j} = 0$

for all $|\alpha|<j$. Now, for any $\zeta \in A$, $\zeta_j = \zeta$ for infinitely many j. Thus, there is a subseq. $z_{j_\ell} \in D_\zeta$ s.t. $\partial^\alpha f(z_{j_\ell})$, $\forall |\alpha| < j_\ell \to \infty$.



If $\exists$ open set $\Omega_2 \supseteq \overline{D_\zeta}$ s.t. f extended holom. to Ω_2, then $\exists$ conv. sub-subseq. $z_{j_{k_n}} \to z \in \Omega_2$ and $\partial^\alpha f(z) = 0$, $\forall \alpha$. This would imply $f \equiv 0$ on a nbhd of z (by P.S. expansion) $\Rightarrow f \equiv 0$ in Ω by uniqueness. This is a contradiction, so f cannot extend holom. to an open nbhd of any $\overline{D_\zeta}$, $\zeta \in A$. But if now Ω_1, Ω_2 as in Def. of DoH existed, then since A is dense, we could clearly find $\zeta \in A \cap \Omega_2$ so close to $\partial\Omega \cap \Omega_2$ s.t. $\overline{D_\zeta} \subseteq \Omega_2$. Thus, Ω is a DoH, by definition. ▨

Thm 4 Let Ω be Reinhard domain w/ $0 \in \Omega$. Then, Ω is a DoH $\Leftrightarrow$ Ω is log-convex.

$\Rightarrow$. is clear from results on Reinhardt domains and DoC.

$\Leftarrow$. See Hormander.